

A KOBAYASHI PSEUDO-DISTANCE FOR HOLOMORPHIC BRACKET GENERATING DISTRIBUTIONS

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ABSTRACT. In this paper, we generalize the Kobayashi pseudo-distance to complex manifolds which admit holomorphic bracket generating distributions. The generalization is based on Chow's theorem in sub-Riemannian geometry. For complex homogeneous manifolds with invariant holomorphic bracket generating distributions, we prove that they have fiberations over flag domains and the fibers are parallelizable complex manifolds.

1. INTRODUCTION

Let M be a complex manifold. We say that D is a holomorphic distribution on M when D is a holomorphic subbundle of the holomorphic tangent bundle T_M of M and we say that it is bracket generating if any local holomorphic frame $\{X_1, \dots, X_d\}$ for D together with all of its iterated Lie brackets $[X_i, X_j]$, $[X_i, [X_j, X_\ell]]$, \dots spans T_M . In this paper, we generalize the Kobayashi pseudo-distance to the complex manifolds which admit a holomorphic bracket generating distribution D on M .

Let us first recall the definition of the Kobayashi pseudo-distance. Let M be a complex manifold and $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ a unit disc. For $x, y \in M$ define

$$\delta_M(x, y) = \inf\{d_\Delta(a, b) : h : \Delta \rightarrow M \text{ holomorphic, } h(a) = x, h(b) = y\}$$

where d_Δ is the Poincaré distance on Δ . The *Kobayashi pseudo-distance* of M is defined by

$$(1.1) \quad d_M(x, y) = \inf \delta_M(x_{j-1}, x_j)$$

where the infimum is taken over all the set of finitely many points $\{x_0, \dots, x_N\}$ in M such that $x = x_0$ and $y = x_N$. When d_M is a distance, M is called *Kobayashi hyperbolic*.

In [12] Kobayashi first introduced it as the largest pseudo-distance on a complex manifold which satisfies the distance decreasing property with respect to the holomorphic mappings between complex manifolds. If a complex manifold has a complete Hermitian metric and its holomorphic sectional curvature is bounded above by a negative constant, then the manifold is complete Kobayashi hyperbolic. Because of this property, complete Kobayashi hyperbolic manifolds are considered as generalizations of the Riemann surfaces of genus greater than or equal to 2 equipped with the Poincaré distance and their study related to those is still active in both several complex variables and complex geometry.

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A holomorphic mapping $f: N \rightarrow M$ between complex manifolds M, N is tangential to D if $df(T_N) \subset D$. The generalization of the Kobayashi pseudo-distance to complex manifolds with holomorphic bracket generating distributions is the following.

Definition 1.1. *Let M be a complex manifold with a holomorphic bracket generating distribution D . For $x, y \in M$, define*

$$(1.2) \quad \delta_{M,D}(x, y) = \inf \{d_{\Delta}(a, b) : h: \Delta \rightarrow M \text{ holomorphic} \\ \text{tangential to } D, h(a) = x, h(b) = y\}$$

where d_{Δ} is the Poincaré distance on Δ . If there is no holomorphic disc tangential to D connecting x and y , then set $\delta_{M,D}(x, y) = \infty$. For $x, y \in M$, define the Kobayashi pseudo-distance of (M, D) by

$$(1.3) \quad d_{M,D}(x, y) = \inf \delta_{M,D}(x_{j-1}, x_j)$$

where the infimum is taken over the set $\{x_0, \dots, x_N\}$ of finitely many points in M such that $x = x_0$ and $y = x_N$. When $d_{M,D}$ is a distance, we say that (M, D) is Kobayashi hyperbolic. Moreover if every Cauchy sequence with respect to $d_{M,D}$ has a convergent subsequence, then we say that (M, D) is complete Kobayashi hyperbolic.

In order to make sense of Definition 1.1, we need the finiteness of $d_{M,D}$ between pair of points belonging to the same connected component.

Theorem 1.2. *Let M be a connected complex manifold with a holomorphic bracket generating distribution D . Then for any two points x, y in M one has $d_{M,D}(x, y) < \infty$.*

This theorem is based on the proof of theorem of Chow given in [7] in the context of sub-Riemannian geometry.

Theorem 1.3 (Chow [7]). *Let M be a connected manifold and D a bracket generating subbundle in the tangent bundle of M . Then any two points in M can be joined by a horizontal piecewise curve.*

In [8] the Kobayashi-Royden infinitesimal pseudo-metric was generalized by Demailly to complex manifolds with holomorphic distributions.

Definition 1.4 (Demailly [8]). *The Kobayashi-Royden infinitesimal pseudo-metric of (M, D) is the Finsler metric on D defined for any $x \in M$ and $v \in D_x$ by*

$$(1.4) \quad k_{M,D}(v) = \inf \{\lambda > 0 : f: \Delta \rightarrow M \text{ holomorphic} \\ \text{tangential to } D, f(0) = x, \lambda df(0) = v\}.$$

We say that (M, D) is infinitesimally Kobayashi hyperbolic if $k_{M,D}$ is positive definite on every fiber D_x and satisfies a uniform lower bound $k_{M,D}(v) \geq \epsilon |v|_g$ for ϵ sufficiently small and depending on any smooth Hermitian metric g on D , when x describes a compact subset of M .

If D is the holomorphic tangent bundle of M itself, then $d_{M,D}$ and $k_{M,D}$ are usual Kobayashi pseudo-distance d_M and Kobayashi-Royden infinitesimal pseudo-metric k_M . By definition

$$(1.5) \quad d_M(x, y) \leq d_{M,D}(x, y), \quad k_M(v) \leq k_{M,D}(v)$$

for any $x, y \in M$ and $v \in D_x$. Hence if the complex manifold M is (infinitesimally) Kobayashi hyperbolic, then (M, D) is also (infinitesimally) Kobayashi hyperbolic. However the converse is not true.

On the other hand suppose that M is a homogeneous complex manifold. In [15], Nakajima proved that if M is Kobayashi hyperbolic, then it is biholomorphic to a Siegel domain of second type, i.e., a bounded homogeneous domain in the Euclidean space. On the other hand if we consider the Kobayashi hyperbolicity for complex manifolds with holomorphic bracket generating distributions, there are more homogeneous examples which are Kobayashi hyperbolic. Let $G^{\mathbb{C}}$ be a complex semisimple Lie group. Let P be a parabolic subgroup in $G^{\mathbb{C}}$ and G a non-compact real form of $G^{\mathbb{C}}$. Then an open G -orbit in the flag manifold G/P is called a *flag domain* (cf. [9]). Suppose that $V := G \cap P$ be a compact subgroup of G containing a maximal compact torus of G . Then we call a flag domain $F = G \cdot z_0 = G/V$ with $z_0 = eP \in G^{\mathbb{C}}/P$ a *canonical flag domain*. Two of the most important examples of canonical flag domains are the period domains and the Mumford–Tate domains. In [5], it is explained that the canonical flag domain F admits a bracket generating holomorphic distribution H_F which is known as the *super-horizontal distribution*. We are able to show that (F, H_F) is complete Kobayashi hyperbolic by exploiting that it has holomorphic sectional curvature bounded above by a negative constant. Under some conditions, a homogeneous complex manifold with a holomorphic bracket generating distribution has a special Lie algebra structure which is similar to that of the canonical flag domains (Theorem 5.1). Furthermore we prove the following.

Theorem 1.5. *Let (M, D) be a Kobayashi hyperbolic complex G -homogeneous manifold with a holomorphic bracket generating G -invariant distribution $D \neq T_M$. Suppose that G acts on M effectively. Then G is a semisimple noncompact real Lie group and M has a holomorphic fibration over a flag domain and the fiber is a parallelizable complex manifold.*

Corollary 1.6. *If G is of inner type, then M is a canonical flag domain.*

The relation between Kobayashi hyperbolicity and the negative curvature is induced from one of the generalizations of Schwarz’s lemma: let $f: \Delta \rightarrow M$ be a holomorphic mapping, where M is a complex manifold equipped with a Hermitian metric g . Suppose that the holomorphic sectional curvature of M is bounded above by a negative constant. Then $f^*g \leq cg_{\Delta}$ for some positive constant c and the Poincaré metric g_{Δ} . Generalization of this Schwarz lemma for the holomorphic discs tangential to the directions for which the holomorphic sectional curvature is bounded above by a negative constant is already well-known (for instance see [6]). In this paper we will consider the generalized Schwarz lemma of Yau given in [19] to obtain the Schwarz lemma given in [6]. Even though we only need the Schwarz lemma given in Corollary 3.2, the generalized Schwarz lemma of Yau for complex manifolds with holomorphic distributions will be described for future reference.

Here is the outline of the paper. In section 2, we recall notions related to holomorphic vector fields and prove Theorem 1.2. The properties are given without proof since they can be obtained through the same proofs used in [12] and in the papers cited. In section 3, we explain the relation between the Kobayashi hyperbolicity of (M, D) and the holomorphic

sectional curvature bounded above by a negative constant. In section 4, we describe canonical flag domains and their super-horizontal distributions. In section 5, we describe the graded Lie algebra structure of homogeneous complex manifolds with holomorphic bracket generating distribution. In particular, we prove 1.5 under the assumption that (M, D) is Kobayashi hyperbolic.

Throughout this paper, the Roman letters a, b, c, \dots run from 1 to m , the Roman letters i, j, k, \dots run from 1 to d , the Greek letters $\alpha, \beta, \gamma, \dots$ run from $d+1$ to m and the Greek letters σ, μ, ν, \dots run from 1 to n . Let $\mathfrak{g}, \mathfrak{v}, \mathfrak{k}, \dots$ denote Lie algebras and G, V, K, \dots the Lie groups corresponding to $\mathfrak{g}, \mathfrak{v}, \mathfrak{k}, \dots$. Given a subspace $\mathfrak{a} \subset \mathfrak{g}$, let $\mathfrak{a}^{\mathbb{C}}$ denote its complexification.

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2. A KOBAYASHI PSEUDO-DISTANCE FOR COMPLEX MANIFOLDS WITH HOLOMORPHIC BRACKET GENERATING DISTRIBUTIONS

2.1. Holomorphic vector fields. Let M be a complex manifold and T_M be the holomorphic tangent bundle of M . A holomorphic vector field X on M is a holomorphic section of T_M . Since T_M is naturally isomorphic to the real tangent bundle TM , we can identify X with a real vector field that we continue to denote by X . A flow ϕ_X associated to X , which is defined on an open subset U of $\mathbb{R} \times M$ containing $\{0\} \times M$ is given in the following way: for $(t, p) \in U$, $\phi_X(t, p) = c_p(t)$, where $c_p: (a(p), b(p)) \rightarrow M$ is the unique maximal solution of the initial value problem

$$(2.1) \quad \begin{aligned} \frac{d}{dt}c(t) &= X \circ c_p(t), \\ c_p(0) &= p. \end{aligned}$$

Let $\phi_X(t)p$ denote $\phi_X(t, p)$. For fixed small t , $p \rightarrow \phi_X(t)p$ is holomorphic and $\{\phi_X(t) : t \in (-\epsilon, \epsilon) \subset \mathbb{R}\}$ is a local one parameter group of holomorphic local diffeomorphisms of M . Define the complex flow of X to be

$$(2.2) \quad \Phi_X(s + it) := \phi_X(s) \circ \phi_{iX}(t).$$

Then Φ_X defines a holomorphic local \mathbb{C} -action.

2.2. Piecewise connected horizontal discs. Let M be a complex manifold of dimension m with a holomorphic bracket generating distribution D of rank d . Fix $x \in M$. Let X_1, \dots, X_d be holomorphic vector fields on a small neighborhood U of x such that $\{X_1|_y, \dots, X_d|_y\}$ is a basis of D_y for all $y \in U$. Choose suitable X_{i_1}, X_{j_1}, \dots such that

$$\begin{aligned} X_{d+1} &= [X_{i_1}, X_{j_1}], \quad X_{d+2} = [X_{i_2}, X_{j_2}], \dots \\ \dots, X_m &= [X_{p_{m-d}}, \dots, [X_{i_{m-d}}, X_{j_{m-d}}], \dots] \end{aligned}$$

form a basis of $T_y M$ with X_1, \dots, X_d at all $y \in U$. Then there is a small open subset $\tilde{U} \subset U$ of x and $W \subset \mathbb{C}^m$ such that the holomorphic mapping $F: W \rightarrow \tilde{U}$ defined by

$$(2.3) \quad F(t_1, \dots, t_m) = \phi_1(t_1) \circ \dots \circ \phi_m(t_m)x$$

is a holomorphic diffeomorphism. Here, ϕ_1, \dots, ϕ_m are the complex flows associated with the vector fields X_1, \dots, X_m that are described in Section 2.1. In particular for each j with $d+1 \leq j \leq m$, ϕ_j is a finite composition of ϕ_i and ϕ_i^{-1} with $1 \leq i \leq d$. Hence for fixed t_1, \dots, t_m , there are holomorphic discs connecting x and $\phi_1(t_1) \circ \dots \circ \phi_m(t_m)x$. Note that

$$\frac{dF}{dt_i}(t_1, \dots, t_m) = d\phi_1(t_1) \circ \dots \circ d\phi_{i-1}(t_{i-1})X_i|_{\phi_{i+1}(t_{i+1}) \circ \dots \circ \phi_m(t_m)x}$$

and in particular $dF|_0 = id$. This implies that there is an open neighborhood of x such that x can be connected to any point in the neighborhood by finite number of holomorphic discs tangential to D . By standard procedure to show the global connectivity using the local connectivity, we can show that if M is connected, every two points in M can be connected by a holomorphic chain of discs that are tangential to D . That is for any $x, y \in M$, we can take a set $\{x_0, \dots, x_N\}$ of finitely many points in M and a set $\{a_1, \dots, a_N\}$ in Δ with $x = x_0$ and $y = x_N$ such that there are holomorphic discs $f_j: \Delta \rightarrow M$ tangential to D for each $j = 1, \dots, N$ with $f_j(0) = x_{j-1}$ and $f_j(a_j) = x_j$. Hence until now we have proved Theorem 1.2.

2.3. Properties of the Kobayashi pseudo-distance and the Kobayashi pseudo-metrics for (M, D) . In this section M, M_1, M_2 are complex manifolds equipped with holomorphic bracket generating distributions D, D_1, D_2 respectively.

Proposition 2.1.

- (1) *The Kobayashi pseudo-distance satisfies the triangle inequality.*
- (2) *Let f be a holomorphic mapping from (M_1, D_1) to (M_2, D_2) such that $df(D_1) \subset D_2$. Then for any $x, y \in M_1$ and $v \in D_1$*

$$(2.4) \quad \begin{aligned} d_{M_2, D_2}(f(x), f(y)) &\leq d_{M_1, D_1}(x, y), \\ k_{M_2, D_2}(df(v)) &\leq k_{M_1, D_1}(v). \end{aligned}$$

Corollary 2.2. *Let G be a complex Lie group acting holomorphically on M . Suppose that D is G -invariant. Then for every $x \in M$, $d_{M, D}(g_1(x), g_2(x)) = 0$ for all $g_1, g_2 \in G$. In particular if (M, D) is Kobayashi hyperbolic, then there is no holomorphic action on M with a complex Lie group which acts D invariant.*

Proposition 2.3 (Demailly [8]). *$k_{M, D}$ is upper semicontinuous on the total space of D . If M is compact, (M, D) is infinitesimally Kobayashi hyperbolic if and only if there are no non-constant entire curves $f: \mathbb{C} \rightarrow M$ tangential to D . In that case, $k_{M, D}$ is a continuous (and positive definite) Finsler metric on D .*

Proposition 2.4 (cf. [16, 1]). *For $x, y \in M$*

$$(2.5) \quad d_{M, D}(x, y) = \inf_{\gamma} \int k_{M, D}(\gamma'(t)) dt$$

where the infimum is taken over all piecewise differentiable curve $\gamma: [0, 1] \rightarrow M$ such that $\gamma'(t) \in D$ for all $t \in [0, 1]$ and $\gamma(0) = x, \gamma(1) = y$.

Corollary 2.5.

- (1) *(M, D) is Kobayashi hyperbolic if and only if (M, D) is infinitesimally Kobayashi hyperbolic.*

(2) The Kobayashi distance $d_{M,D}$ is continuous.

Proposition 2.6. Let \widetilde{M} be a covering manifold of M with covering projection $\pi: \widetilde{M} \rightarrow M$. Define a holomorphic bracket generating distribution \widetilde{D} on \widetilde{M} by $d\pi^{-1}(D)$.

(1) Let $x, y \in M$ and choose $\tilde{x} \in \widetilde{M}$ such that $\pi(\tilde{x}) = x$. Then

$$(2.6) \quad d_{M,D}(x, y) = \inf_{\tilde{y}} d_{\widetilde{M}, \widetilde{D}}(\tilde{x}, \tilde{y})$$

where the infimum is taken over all $\tilde{y} \in \widetilde{M}$ such that $y = \pi(\tilde{y})$.

(2) (M, D) is (complete) Kobayashi hyperbolic if and only if $(\widetilde{M}, \widetilde{D})$ is (complete) Kobayashi hyperbolic.

Proposition 2.7. Let $\pi: P \rightarrow M$ be a holomorphic fiber bundle over M . Suppose that (M, D) is (complete) Kobayashi hyperbolic. Let $T_P = \mathcal{H} \oplus \mathcal{V}$ be a decomposition where \mathcal{V} is the vertical distribution and \mathcal{H} is a horizontal distribution (i.e., $d\pi|_{\mathcal{H}}: \mathcal{H} \rightarrow T_M$ is an isomorphism). Set $\widetilde{D} = (d\pi|_{\mathcal{H}})^{-1}(D) \subset \mathcal{H}$.

(1) Suppose that \widetilde{D} is bracket generating in T_P . Then (P, \widetilde{D}) is (complete) Kobayashi hyperbolic.

(2) Suppose that the fiber of P is (complete) Kobayashi hyperbolic. Then $(P, \widetilde{D} \oplus \mathcal{V})$ is (complete) Kobayashi hyperbolic.

Proposition 2.8. Suppose that there is an embedding $\iota: M_1 \rightarrow M_2$ such that $d\iota(D_1) \subset D_2$. Then if (M_2, D_2) is (complete) hyperbolic, then (M_1, D_1) is also (complete) hyperbolic.

Proposition 2.9. Suppose that (M, D) is complete Kobayashi hyperbolic. Then the set of holomorphic mappings from M_1 to (M, D) tangential to D is a normal family; i.e., every sequence of holomorphic mappings from M_1 to M tangential to D either has a subsequence that converges uniformly on compact subsets or has a compactly divergent subsequence.

3. KOBAYASHI HYPERBOLICITY AND NEGATIVE CURVATURE

3.1. Generalized Schwarz's lemma for (M, D) and Kobayashi hyperbolicity for (M, D) with negative holomorphic sectional curvatures. Let M be a complex manifold of complex dimension m and D a holomorphic distribution on M of rank d with a Hermitian metric g . Let $A^k(D)$ denote the set of D -valued k -forms on M . Given a Hermitian metric g , there is a unique connection, called the Chern connection, $\nabla: A^0(D) \rightarrow A^1(D)$ compatible with both the complex structure of M and the metric g . Choose a local orthonormal frame e_1, \dots, e_d of D and denote its dual frame by $\theta_1, \dots, \theta_d$, i.e., $g = \sum \theta_i \bar{\theta}_i$. Denote θ_{ij} a connection 1-form, i.e.,

$$\nabla e_i = \sum \theta_{ij} \otimes e_j.$$

If one extends g to a Hermitian metric of T_M and denotes $T_M = D \oplus D^\perp$ where D^\perp is the orthogonal complement of D in T_M , then $\nabla = \pi_D \circ \nabla_M$ where ∇_M is the Chern connection with respect to the extended Hermitian metric on T_M and π_D is the orthogonal projection onto D . By adding a local frame of D^\perp we complete the frame to e_1, \dots, e_m and the dual frame to $\theta_1, \dots, \theta_m$. Then $d\theta_i - \theta_{ij} \wedge \theta_j$ can be expressed by

$$(3.1) \quad d\theta_i = \theta_{ij} \wedge \theta_j + \Theta_i + N_i$$

where N_i contains θ_α or $\bar{\theta}_\alpha$ with $d+1 \leq \alpha \leq m$ and Θ_i consists of θ_j and $\bar{\theta}_j$ with $1 \leq j \leq d$. We can express

$$(3.2) \quad \Theta_i = \Theta_i^{2,0} + \Theta_i^{1,1} + \Theta_i^{0,2}$$

with a (a, b) -form $\Theta_i^{a,b}$. Note that $\Theta_i^{0,2} = 0$ and $\Theta_i^{1,1} = 0$ since M is a complex manifold and ∇ is the Chern connection. The curvature tensor is given by

$$(3.3) \quad \Theta_{ij} = d\theta_{ij} - \theta_{ik} \wedge \theta_{kj} = R_{ij\bar{a}b} \theta_a \wedge \bar{\theta}_b.$$

For $X = \sum X_i e_i$, $Y = \sum Y_i e_i \in D_p$, define a holomorphic bisectonal curvature of (M, D, g) in directions X and Y by

$$(3.4) \quad \text{Bisec}_{(D,g)}(X, Y) = \frac{\sum R_{ijk\bar{l}} X_i \bar{X}_j Y_k \bar{Y}_l}{\sum |X_i|^2 \sum |Y_i|^2}$$

and the holomorphic sectional curvature of (M, D, g) in direction X by

$$(3.5) \quad H_{(D,g)}(X) = \text{Bisec}_{(D,g)}(X, X).$$

For a complex manifold N equipped with a Hermitian metric h , let $\{\omega_\sigma\}$ is an orthonormal frame of (N, h) . For a holomorphic mapping $f: N \rightarrow M$ tangential to D , let $u = \sum_{i\sigma} a_{i\sigma} \bar{a}_{i\sigma}$ where $f^*(\theta_i) = \sum_\sigma a_{i\sigma} \omega_\sigma$. Then by the Omori-Yau maximum principle and the same calculations of Laplacian of u in [19], we have the following theorem.

Theorem 3.1. *Let (N, h) be a complete Hermitian manifold with Ricci curvature bounded from below by a constant K_1 . Let (M, D, g) be a complex manifold with a holomorphic distribution D equiped with a Hermitian metric g . Suppose that the holomorphic bisectonal curvature of (M, D, g) is bounded from above by a negative constant K_2 . Then for any holomorphic mapping f from N into M tangential to D , we have*

$$(3.6) \quad f^*(g) \leq \frac{K_1}{K_2} h.$$

Corollary 3.2 (cf. [6]). *Let (Δ, g_Δ) be the unit disc in \mathbb{C} with the Poincaré metric g_Δ with curvature -1 . Let (M, D, g) be a complex manifold with a holomorphic distribution D and g a Hermitian metric on D . Suppose that the holomorphic sectional curvature of (M, D, g) is bounded from above by a negative constant K . Then for any holomorphic mapping f from Δ into M tangential to D , we have*

$$(3.7) \quad f^*(g) \leq \frac{1}{K} g_\Delta.$$

Define for any $x, y \in M$

$$(3.8) \quad \rho_{M,D,g}(x, y) = \inf_\gamma \int_\gamma g(\gamma'(t), \gamma'(t))^{1/2} dt$$

where the infimum is taken over all $\gamma: [0, 1] \rightarrow M$ tangential to D such that $\gamma(0) = x$, $\gamma(1) = y$. If there is no curve with this condition, define $\rho_{M,D,g}(x, y) = \infty$. $\rho_{M,D,g}$ is called the *Carnot-Caratheodory distance* with respect to (M, D, g) .

Theorem 3.3 (cf. [14]). *Suppose that D is bracket generating. Then $\rho_{M,D,g}$ is finite, continuous on M and induces the manifold topology.*

Let $\rho_{N,h}$ be a distance function of h on N . Then Theorem 3.1 implies that for $a, b \in N$,

$$(3.9) \quad \rho_{M,D,g}(f(a), f(b)) \leq \sqrt{\frac{K_1}{K_2}} \rho_{N,h}(a, b).$$

If the holomorphic sectional curvature of (M, D, g) is bounded by a negative constant, for any holomorphic mapping $f: \Delta \rightarrow M$ tangential to D , $f^*g \leq cg_\Delta$ for some positive constant c by Corollary 3.2. This implies that for any $a, b \in \Delta$

$$\rho_{M,D,g}(f(a), f(b)) \leq \sqrt{c} \rho_\Delta(a, b).$$

Suppose that $d_{M,D}(x, y) = 0$ for some $x, y \in M$. This implies that there are sequence $f_i: \Delta \rightarrow M$, $a_i, b_i \in \Delta$ such that $f(a_i) = x$, $f(b_i) = y$ and $\rho_\Delta(a_i, b_i) \rightarrow 0$ as $i \rightarrow \infty$. Hence $\rho_{M,D,g}(x, y) = 0$ and hence $x = y$ by Theorem 3.3. Hence we obtain the following.

Corollary 3.4. *Suppose that holomorphic sectional curvature of (M, D, g) is bounded from above by a negative constant. Then (M, D) is Kobayashi hyperbolic. If the sub-Riemannian distance is complete, then the Kobayashi distance of (M, D) is also complete.*

Remark 3.5. *Suppose that D is a subbundle of the holomorphic tangent bundle of M . Here D does not need to be holomorphic bracket generating. Let g be a Hermitian metric on the tangent bundle of M and let Θ denote the Chern curvature tensor of g on M . If $\frac{g(\Theta(\zeta, \xi)\zeta, \xi)}{g(\zeta, \zeta)g(\xi, \xi)}$ is bounded above by a negative constant for all $\zeta, \xi \in D$, then the same argument in Theorem 3.1 holds. That is, for every holomorphic mapping $f: N \rightarrow M$ tangential to D , $f^*(g) \leq \frac{K_1}{K_2} h$. Furthermore Corollary 3.2 also holds as 13.4.1.Schwarz's lemma given in [6].*

4. EXAMPLES OF KOBAYASHI HYPERBOLIC MANIFOLD (M, D) :

CANONICAL FLAG DOMAINS WITH SUPER-HORIZONTAL DISTRIBUTIONS

In this section, we only consider the canonical flag domains. Let T denote a maximal torus, $V = G \cap P$ of G at z_0 the isotropy group and K the unique maximal compact subgroup of G containing V . Let $\mathfrak{t} \subset \mathfrak{v} \subset \mathfrak{k} \subset \mathfrak{g}$ be Lie algebras of $T \subset V \subset K \subset G$ respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ be the Cartan decomposition. The Killing form B of \mathfrak{g} is negative definite on \mathfrak{k} and positive definite on \mathfrak{q} . Then $\mathfrak{h} = \mathfrak{t}^\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}^\mathbb{C}$. Let $\Phi = \Phi(\mathfrak{g}^\mathbb{C}, \mathfrak{h})$ denote the set of roots of $\mathfrak{g}^\mathbb{C}$ respect to \mathfrak{h} . Given a root $\alpha \in \Phi$, let $\mathfrak{g}^\alpha \subset \mathfrak{g}^\mathbb{C}$ denote the associated root space. Given a subspace $\mathfrak{s} \in \mathfrak{g}^\mathbb{C}$, define

$$\Phi(\mathfrak{s}) = \{\alpha \in \Phi: \mathfrak{g}^\alpha \subset \mathfrak{s}\}.$$

We will call the root $\alpha \in \Phi(\mathfrak{k}^\mathbb{C})$ compact and $\alpha \in \Phi(\mathfrak{q}^\mathbb{C})$ non-compact. Fix a Borel subalgebra $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}^\mathbb{C}$. Then

$$\Phi^+ = \Phi(\mathfrak{b})$$

determines a set of positive roots and let Φ^- denote the set of corresponding negative roots. Given a subspace $\mathfrak{s} \subset \mathfrak{g}^\mathbb{C}$, define

$$\Phi^+(\mathfrak{s}) = \Phi(\mathfrak{s}) \cap \Phi^+, \quad \Phi^-(\mathfrak{s}) = \Phi(\mathfrak{s}) \cap \Phi^-$$

Let $\{\alpha_1, \dots, \alpha_r\} \subset \Phi^+$ denote the corresponding simple roots. For each $\alpha \in \Phi$, one can choose vectors $e_\alpha \in \mathfrak{g}^\alpha$ and $h_\alpha \in \mathfrak{h}$ such that

$$(1) \quad B(e_\alpha, e_\beta) = \delta_{\alpha, -\beta} \quad \text{and} \quad [e_\alpha, e_{-\alpha}] = h_\alpha,$$

- (2) $B(h_\alpha, x) = \alpha(x)$ for every $x \in \mathfrak{h}$,
- (3) $[e_\alpha, e_\beta] = 0$ if $\alpha \neq -\beta$ and $\alpha + \beta \notin \Phi$,
- (4) $[e_\alpha, e_\beta] = N_{\alpha, \beta} e_{\alpha+\beta}$ if $\alpha, \beta, \alpha + \beta \in \Phi$ where $N_{\alpha, \beta}$ are nonzero real constants such that

$$N_{-\alpha, -\beta} = -N_{\alpha, \beta}, \quad N_{-\alpha, -\beta} = N_{-\beta, \alpha+\beta} = N_{\alpha+\beta, -\alpha},$$

- (5) For the complex conjugate σ of \mathfrak{g} in $\mathfrak{g}^\mathbb{C}$, $\sigma(e_\alpha) = \epsilon_\alpha e_{-\alpha}$ where $\epsilon_\alpha = -1$ if α is compact, $\epsilon_\alpha = 1$ if α is non-compact,

Let ω^α be dual covectors in $(\mathfrak{g}^\mathbb{C})^*$ of $e_\alpha \in \mathfrak{g}^\alpha$. Let $\{T^1, \dots, T^r\}$ be the basis of \mathfrak{h} dual to the simple roots $\{\alpha_1, \dots, \alpha_r\}$. Define $T = \sum_{\alpha \in \Phi \setminus \Phi(\mathfrak{v}^\mathbb{C})} T^i$. Then $\mathfrak{g}^\mathbb{C}$ has the decomposition

$$(4.1) \quad \mathfrak{g}^\mathbb{C} = \mathfrak{g}_k \oplus \dots \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \dots \oplus \mathfrak{g}_{-k}$$

where $\mathfrak{g}_l = \{\zeta \in \mathfrak{g}^\mathbb{C} : [T, \zeta] = l\zeta\}$. Remark that

$$(4.2) \quad \mathfrak{g}_0 = \mathfrak{v}^\mathbb{C}, \quad \mathfrak{k}^\mathbb{C} = \sum_{i \text{ even}} \mathfrak{g}_i \quad \text{and} \quad \mathfrak{q}^\mathbb{C} = \sum_{i \text{ odd}} \mathfrak{g}_i.$$

Denote $\mathfrak{g}_+ = \bigoplus_{j>0} \mathfrak{g}_j$ and $\mathfrak{g}_- = \bigoplus_{j<0} \mathfrak{g}_j$. Then a homogeneous complex structure on F is given by specifying $T^\mathbb{C}F = TF \otimes \mathbb{C} = T^{1,0}F \oplus T^{0,1}F$ with $T^{1,0}F = G \times_V \mathfrak{g}_-$, $T^{0,1}F = G \times_V \mathfrak{g}_+$ where $T^{1,0}D$ is the holomorphic tangent bundle of D . Let

$$(4.3) \quad H_F := G \times_V \mathfrak{g}_{-1}$$

denote the super-horizontal distribution given in [5]. Note that if F is a Hermitian symmetric space of non-compact type, then H_F is the holomorphic tangent bundle T_F . Let g be a G -invariant Hermitian metric on H_F defined by

$$(4.4) \quad g(\zeta, \xi) = B(\zeta, \sigma(\xi)).$$

In [11] Griffiths and Schmid calculated the curvature tensor which is expressed by

$$(4.5) \quad \begin{aligned} \Theta_F = - \sum_{\alpha, \beta \in \Phi^+ \setminus \Phi^+(\mathfrak{k}^\mathbb{C})} \text{ad}_{[e_\alpha, e_{-\beta}]_{\mathfrak{v}^\mathbb{C}}} \otimes \omega^\alpha \wedge \bar{\omega}^\beta \\ + \sum_{\alpha, \beta \in \Phi^+(\mathfrak{k}^\mathbb{C}) \setminus \Phi^+(\mathfrak{v}^\mathbb{C})} \text{ad}_{[e_\alpha, e_{-\beta}]_{\mathfrak{v}^\mathbb{C}}} \otimes \omega^\alpha \wedge \bar{\omega}^\beta \end{aligned}$$

where $[e_\alpha, e_{-\beta}]_{\mathfrak{v}^\mathbb{C}}$ denote the projection of $[e_\alpha, e_{-\beta}]$ onto $\mathfrak{v}^\mathbb{C}$. For $\zeta, \xi \in H_F$ at eV , the holomorphic bisectional curvature of (F, H_F, g) is given by

$$(4.6) \quad \text{Bisec}_{F, H_F}(\zeta, \xi) = - \frac{B([\zeta, \sigma(\zeta)], \xi, \sigma(\xi))}{g(\zeta, \zeta)g(\xi, \xi)} = - \frac{B([\zeta, \sigma(\zeta)], [\xi, \sigma(\xi)])}{g(\zeta, \zeta)g(\xi, \xi)}$$

Let $\zeta \in \mathfrak{g}_{-1}$. then $[\zeta, \sigma(\zeta)] \in \sqrt{-1}\mathfrak{v}$. Note that B is positive definite on $\sqrt{-1}\mathfrak{v}$. Hence there should be a positive constant c such that

$$(4.7) \quad H_{F, H_F}(\zeta) = - \frac{B([\zeta, \sigma(\zeta)], [\zeta, \sigma(\zeta)])}{g(\zeta, \zeta)^2} < -c < 0$$

for all $\zeta \in H_F$. By Corollary 3.4, homogeneity of F and the ball-box theorem given in [14], we obtain the following.

Theorem 4.1. *Canonical flag domains with super-horizontal distributions are complete Kobayashi hyperbolic.*

Corollary 4.2. *Let Γ be an uniform lattice of G , that is $\Gamma \backslash G$ is compact. Then $\Gamma \backslash F$ with the distribution induced from the super-horizontal distribution of F is compact and Kobayashi hyperbolic.*

Suppose that F is not a Hermitian symmetric space of non-compact type. Then by the Matsuki correspondence, there exists a unique $K^\mathbb{C}$ -orbit in F and this $K^\mathbb{C}$ -orbit which is called *base cycle*, is a compact complex submanifold of F . This implies that F is not Kobayashi hyperbolic

Corollary 4.3. *There is no holomorphic embedding from a flag domain which is not Hermitian symmetric into the canonical flag domain tangential to the super-horizontal distribution.*

Proof. Let F_1, F_2 be flag domains where F_1 is not a Hermitian symmetric space and D_2 the super-horizontal distribution of F_2 . Let $\iota: F_1 \rightarrow F_2$ be a holomorphic embedding tangential to D_2 . Then by Proposition 2.8, $d_{F_2, D_2}(\iota(x), \iota(y)) \leq d_{F_1}(x, y)$. Suppose that $x \neq y$ are in the base cycle of F_1 . Then $d_{F_1}(x, y) = 0$ and this induces the contradiction to that (F_2, D_2) is Kobayashi hyperbolic. \square

Remark 4.4. *In [6], they proved that the Chern connection on T_F with the same Hermitian metric (4.4) has holomorphic sectional curvature bounded above by a negative constant in non-compact direction $\mathfrak{q}^\mathbb{C}$. Let $\iota: F_1 \rightarrow F_2$ be a holomorphic embedding tangential to D_2 . Then we can obtain that $\rho_{F_2, T_{F_2}, g}(\iota(x), \iota(y)) \leq \rho_{F_2, D_2, g}(\iota(x), \iota(y)) \leq cd_{F_1}(x, y)$ for a positive constant c by Remark 3.5. Hence there is no holomorphic embedding from a flag domain which is not Hermitian symmetric into the canonical flag domain tangential to $\mathfrak{q}^\mathbb{C}$ -direction.*

5. KOBAYASHI HYPERBOLIC HOMOGENEOUS MANIFOLDS

5.1. General Lie algebra structures on complex homogeneous manifolds with holomorphic bracket generating distributions. Let M be a complex homogeneous manifold under the holomorphic action of a Lie group G and D a holomorphic bracket generating distribution on M which is invariant with respect to the action of G . In short, we will say M is a complex G -homogeneous manifold and D is a holomorphic bracket generating G -invariant distribution. Let V denote the isotropy subgroup of G at $x_0 \in M$ and hence M is biholomorphic to G/V . Suppose that V is compact. Let \mathfrak{g} denote the Lie algebra of G and \mathfrak{v} the Lie algebra of V in \mathfrak{g} . Then there are vector subspace \mathfrak{m} of \mathfrak{g} and a decomposition

$$(5.1) \quad \mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m} \quad \text{such that} \quad [\mathfrak{v}, \mathfrak{m}] \subset \mathfrak{m}.$$

The tangent space of M at x_0 is equivalent to $\mathfrak{g}/\mathfrak{v} = \mathfrak{m}$. Let $\mathfrak{g}_1^\mathbb{R}$ denote the subspace of \mathfrak{m} corresponding to D_{x_0} . Define

$$(5.2) \quad \mathfrak{g}_2^\mathbb{R} = [\mathfrak{g}_1^\mathbb{R}, \mathfrak{g}_1^\mathbb{R}]/(\mathfrak{v} + \mathfrak{g}_1^\mathbb{R}), \dots, \mathfrak{g}_k^\mathbb{R} = [\mathfrak{g}_1^\mathbb{R}, \mathfrak{g}_{k-1}^\mathbb{R}]/(\mathfrak{v} + \mathfrak{g}_1^\mathbb{R} + \dots + \mathfrak{g}_{k-1}^\mathbb{R}).$$

Then we can express

$$(5.3) \quad \mathfrak{g} = \mathfrak{v} \oplus \mathfrak{g}_1^\mathbb{R} \oplus \dots \oplus \mathfrak{g}_k^\mathbb{R}.$$

Since $M = G/V$ is a complex manifold and D is G -invariant with compact V , without loss of generality we may assume that there is an endomorphism $j: \mathfrak{g} \rightarrow \mathfrak{g}$ which is induced from the complex structure of M such that

$$(5.4) \quad j\mathfrak{v} = 0, j^2x = -x,$$

$$(5.5) \quad \text{Ad}_v jx = j \circ \text{Ad}_v x,$$

$$(5.6) \quad [jx, jy] = [x, y] + j[jx, y] + j[x, jy],$$

$$(5.7) \quad \text{Ad}_v \mathfrak{g}_1^{\mathbb{R}} \subset \mathfrak{g}_1^{\mathbb{R}},$$

where $x, y \in \mathfrak{g}$ and $v \in V$. Extend j complex linearly to the complexification $\mathfrak{g}^{\mathbb{C}}$ of \mathfrak{g} and denote \mathfrak{g}^+ , \mathfrak{g}^- the eigenspaces of j mod $\mathfrak{v}^{\mathbb{C}}$ with eigenvalues $\sqrt{-1}$, $-\sqrt{-1}$ respectively. Then \mathfrak{g}^+ and \mathfrak{g}^- represent the holomorphic tangent bundle and anti-holomorphic tangent bundle of M respectively. Let \mathfrak{g}_1 denote the subspace in \mathfrak{g}^+ corresponding to D_{x_0} . Then $\mathfrak{g}_1 \cap \mathfrak{g} = \mathfrak{g}_1^{\mathbb{R}}$ and for each $x \in \mathfrak{g}_1^{\mathbb{R}}$, $x - \sqrt{-1}jx \in \mathfrak{g}_1$. Define

$$(5.8) \quad \mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1]/(\mathfrak{v}^{\mathbb{C}} + \mathfrak{g}_1), \dots, \mathfrak{g}_k = [\mathfrak{g}_1, \mathfrak{g}_{k-1}]/(\mathfrak{v}^{\mathbb{C}} + \mathfrak{g}_1 + \dots + \mathfrak{g}_{k-1}).$$

Then

$$(5.9) \quad \mathfrak{g}^+ = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$$

and by the similar way we can construct $\mathfrak{g}_{-1}, \dots, \mathfrak{g}_{-k}$ and

$$(5.10) \quad \mathfrak{g}^- = \mathfrak{g}_{-1} \oplus \dots \oplus \mathfrak{g}_{-k}.$$

For the notational convention, denote

$$(5.11) \quad \mathfrak{g}_0 := \mathfrak{v}^{\mathbb{C}}, \quad \mathfrak{g}_{\leq l} := \sum_{j \leq l} \mathfrak{g}_j \quad \text{and} \quad \mathfrak{g}_{\geq l} := \sum_{j \geq l} \mathfrak{g}_j.$$

Since $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$, we obtain $[\mathfrak{g}_0, \mathfrak{g}_\ell] \subset \mathfrak{g}_{\leq \ell}$ for each $2 \leq \ell \leq k$.

Since D is a holomorphic distribution, it is well-known that

$$(5.12) \quad [\mathfrak{g}_1, \mathfrak{g}^-] \subset \mathfrak{g}_0 + \mathfrak{g}^-.$$

Theorem 5.1. *Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$ be the graded Lie algebra structure defined above with the condition (5.12). Then*

- (1) $[\mathfrak{g}_i, \mathfrak{g}_{-\ell}] \subset \mathfrak{g}_{\leq i-\ell}$ when $i \geq \ell$,
- (2) $[\mathfrak{g}_i, \mathfrak{g}_{-\ell}] \subset \mathfrak{g}_{\geq i-\ell}$ when $i \leq \ell$,
- (3) $[\mathfrak{g}_i, \mathfrak{g}_{-i}] \subset \mathfrak{g}_0$.

Proof. It is enough to prove (1). Since $[\mathfrak{g}_1, \mathfrak{g}_{-1}] \subset \mathfrak{g}^- + \mathfrak{g}_0$, by taking complex conjugation, we obtain that $[\mathfrak{g}_1, \mathfrak{g}_{-1}] \subset \mathfrak{g}_0$. Suppose that for every $i < i'$, $[\mathfrak{g}_i, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{\leq i-1}$. Then $[\mathfrak{g}_{i'}, \mathfrak{g}_{-1}] = [\mathfrak{g}_1, [\mathfrak{g}_{i'-1}, \mathfrak{g}_{-1}]] + [\mathfrak{g}_{i'-1}, [\mathfrak{g}_1, \mathfrak{g}_{-1}]] \subset \mathfrak{g}_{\leq i'-1}$ and hence for every i ,

$$[\mathfrak{g}_i, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{\leq i-1}.$$

Fix i' and suppose that for every $i < i'$, $[\mathfrak{g}_i, \mathfrak{g}_{-\ell}] \subset \mathfrak{g}_{\leq i-\ell}$ for all $1 \leq \ell \leq i$. Furthermore for fixed $\ell' \leq i'$ suppose that $[\mathfrak{g}_{i'}, \mathfrak{g}_{-\ell}] \subset \mathfrak{g}_{\leq i'-\ell}$ for all $\ell < \ell'$. Then since $[\mathfrak{g}_{i'}, \mathfrak{g}_{-\ell'}] = [[\mathfrak{g}_{i'}, \mathfrak{g}_{-\ell'+1}], \mathfrak{g}_{-1}] + [[\mathfrak{g}_{i'}, \mathfrak{g}_{-1}], \mathfrak{g}_{-\ell'+1}]$, we obtain that $[\mathfrak{g}_{i'}, \mathfrak{g}_{-\ell'}] \subset \mathfrak{g}_{\leq i'-\ell'}$ and the Lemma is also proved. \square

Corollary 5.2. *Assume the same condition of Theorem 5.1.*

- (1) For $x \in \mathfrak{g}^{\mathbb{R}}$ and $X = x - \sqrt{-1}jx$, $[X, \overline{X}] \in \sqrt{-1}\mathfrak{v}$ and $[x, jx] \in \mathfrak{v}$.
 (2) Suppose that \mathfrak{g} is semisimple. Then $[\mathfrak{g}_1, \mathfrak{g}_{-1}] = \mathfrak{v}^{\mathbb{C}}$.

Proof. (1) For $x \in \mathfrak{g}^{\mathbb{R}}$ and $X := x - \sqrt{-1}jx$, we obtain $[X, \overline{X}] = [x - \sqrt{-1}jx, x + \sqrt{-1}jx] = 2\sqrt{-1}[x, jx]$. This implies (1).

(2) By (1), $[X, \overline{X}] \in \sqrt{-1}\mathfrak{v}$ for every $X \in \mathfrak{g}_1$, we can obtain that $[X, \overline{Y}] \in \mathfrak{v}^{\mathbb{C}}$ for every $X, Y \in \mathfrak{g}_1$. Let B be the Killing form of $\mathfrak{g}^{\mathbb{C}}$ and then $B_{\tau}(X, Y) = -B(X, \tau(Y))$ be a positive definite Hermitian form on $\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$ with complex conjugation τ with respect to the maximal compact subgroup of $G^{\mathbb{C}}$. Hence B_{τ} is positive definite on $\mathfrak{g}_1 \times \mathfrak{g}_1$. Suppose that there is $v \in \mathfrak{v}^{\mathbb{C}}$ such that $B_{\tau}(v, [x_1, x_2]) = 0$ for all $x_1 \in \mathfrak{g}_1, x_2 \in \mathfrak{g}_{-1}$. Since

$$0 = B_{\tau}(v, [x_1, x_2]) = -B(v, [x_1, x_2]) = B([x_1, v], x_2),$$

$[x_1, v] = 0$ for all $x_1 \in \mathfrak{g}_1$. This implies that $\text{Ad}_{\exp tv} = \text{id}$ on \mathfrak{g}_1 for all $t \in \mathbb{R}$. By the bracket generating property of \mathfrak{g}_1 , $\text{Ad}_{\exp tv} = \text{id}$ on $\mathfrak{g}_{\geq 0}$. But the isotropy representation has finite kernel, $\exp tv = e$ for all t . Hence $v = 0$. \square

5.2. Kobayashi hyperbolic complex homogeneous manifold with holomorphic bracket generating distribution. Let M be a G -homogeneous complex manifold with holomorphic bracket generating G -invariant distribution D on M . Suppose that (M, D) is Kobayashi hyperbolic. Then by Corollary 2.2

$$(5.13) \quad [x, jx] \neq 0 \text{ for all } x \in \mathfrak{g}_1^{\mathbb{R}}.$$

Suppose that G acts on M effectively. Then for the maximal solvable ideal $\mathfrak{r} \subset \mathfrak{g}$,

$$(5.14) \quad \mathfrak{r} \cap \mathfrak{v} = 0.$$

Lemma 5.3. \mathfrak{g} is semisimple.

Proof. Let \mathfrak{r}_1 denote $\mathfrak{g}_1 \cap \mathfrak{r}$. Then it is known (cf. [4]) that $\mathfrak{r} = [\mathfrak{g}, \mathfrak{g}]^{\perp}$ where $[\mathfrak{g}, \mathfrak{g}]^{\perp}$ is the orthogonal complement with respect to the Killing form on \mathfrak{g} and $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r} = [\mathfrak{g}, \mathfrak{r}]$. Hence by bracket generating property, it is enough to prove that $\mathfrak{r}_1 = 0$.

Suppose that $\mathfrak{r}_1 \neq 0$ and let r be a non-zero element in \mathfrak{r}_1 . Then by (5.13) and Corollary 5.2 (1), $[r - \sqrt{-1}jr, r + \sqrt{-1}jr] = 2\sqrt{-1}[r, jr] \in \mathfrak{v}$. On the other hand since \mathfrak{r} is an ideal $[r, jr] \in \mathfrak{r}$ and hence it is a contradiction to (5.14). \square

From now on without loss of generality by Lemma 5.3, assume that \mathfrak{g} is non-compact real form of complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$.

Let \mathfrak{k} be a maximal compact Lie subalgebra in \mathfrak{g} containing \mathfrak{v} . Note that there could be several maximal compact Lie subalgebras containing \mathfrak{v} . Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{q}$ be a Cartan decomposition and θ the Cartan involution which is defined by $k + q \mapsto k - q$ for all $k \in \mathfrak{k}$ and $q \in \mathfrak{q}$. Note that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{q}] \subset \mathfrak{q}$.

Definition 5.4. If $\mathfrak{g}_1^{\mathbb{R}}$ is invariant with respect to a Cartan involution, we say that D is Cartan invariant.

If D is Cartan invariant with respect to a Cartan involution $\theta: \mathfrak{g} \rightarrow \mathfrak{g}$, then $\mathfrak{g}_1^{\mathbb{R}}$ can be decomposed as $\mathfrak{g}_1^{\mathbb{R}} = \mathfrak{g}_1^{\mathbb{R}} \cap \mathfrak{k} \oplus \mathfrak{g}_1^{\mathbb{R}} \cap \mathfrak{q}$ where \mathfrak{k} and \mathfrak{q} are the eigenspaces of eigenvalues 1 and -1 respectively. In case (M, D) is a canonical flag domain with the super-horizontal distribution, we know the relation between the graded Lie algebra structure and the

Cartan decomposition (4.2). Similarly when the distribution is Cartan invariant we obtain the following relation.

Theorem 5.5. *Suppose that (M, D) is a Kobayashi hyperbolic complex G -homogeneous manifold and D is G -invariant. Then G is semisimple Lie group of non-compact type. Furthermore if D is Cartan invariant, then*

$$(5.15) \quad \mathfrak{k} = \mathfrak{v} + \sum_{j \text{ even}} \mathfrak{g}_j^{\mathbb{R}}, \quad \mathfrak{p} = \sum_{j \text{ odd}} \mathfrak{g}_j^{\mathbb{R}} \mod \mathfrak{v}$$

Proof. By Lemma 5.3, we obtain that \mathfrak{g} is a semisimple Lie group of non-compact type. Suppose that $\mathfrak{k}_1 := \mathfrak{k} \cap \mathfrak{g}_1^{\mathbb{R}} \neq 0$. Define a Lie subalgebra

$$\mathfrak{k}' := \mathfrak{v} \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \cdots \oplus \mathfrak{k}_k$$

where $\mathfrak{k}_2 = [\mathfrak{k}_1, \mathfrak{k}_1]/\mathfrak{v}, \dots, \mathfrak{k}_j = [\mathfrak{k}_1, \mathfrak{k}_{j-1}]/(\mathfrak{v} + \mathfrak{k}_0 + \cdots + \mathfrak{k}_{j-1})$. Let K' denote the connected Lie subgroup of G respect to the Lie algebra \mathfrak{k}' . Then $(K'/V, D')$ with holomorphic distribution D' corresponding to \mathfrak{k}_1 is Kobayashi hyperbolic. However since K'/V is a compact homogeneous complex manifold, it cannot be Kobayashi hyperbolic. Hence \mathfrak{k}_1 should be 0. This implies that $\mathfrak{g}_1^{\mathbb{R}} \subset \mathfrak{p}$. Furthermore \mathfrak{g} has a graded Lie algebra structure

$$(5.16) \quad \mathfrak{g} = \mathfrak{v} \oplus \mathfrak{g}_1^{\mathbb{R}} \oplus \mathfrak{g}_2^{\mathbb{R}} \oplus \cdots \oplus \mathfrak{g}_k^{\mathbb{R}}$$

where $\mathfrak{g}_2^{\mathbb{R}} = [\mathfrak{g}_1^{\mathbb{R}}, \mathfrak{g}_1^{\mathbb{R}}]/\mathfrak{v}$, $\mathfrak{g}_3^{\mathbb{R}} = [\mathfrak{g}_1^{\mathbb{R}}, \mathfrak{g}_2^{\mathbb{R}}]/(\mathfrak{v} + \mathfrak{g}_1^{\mathbb{R}})$, and $\mathfrak{g}_{2j+1}^{\mathbb{R}} = [\mathfrak{g}_1^{\mathbb{R}}, \mathfrak{g}_{2j}^{\mathbb{R}}]/(\mathfrak{v} + \mathfrak{g}_1^{\mathbb{R}} + \mathfrak{g}_3^{\mathbb{R}} + \cdots + \mathfrak{g}_{2j-1}^{\mathbb{R}})$, $\mathfrak{g}_{2j}^{\mathbb{R}} = [\mathfrak{g}_1^{\mathbb{R}}, \mathfrak{g}_{2j-1}^{\mathbb{R}}]/(\mathfrak{v} + \mathfrak{g}_2^{\mathbb{R}} + \cdots + \mathfrak{g}_{2j-2}^{\mathbb{R}})$ with (5.15). \square

A complex manifold X is called *parallelizable* if its holomorphic tangent bundle is trivial. In [18], Wang proved that a connected compact complex manifold X is parallelizable if and only if X is homogeneous and X is expressed in H/Γ where H is a connected complex Lie group and Γ is a uniform discrete subgroup of H . For the proof of the theorem, we need the following theorem.

Theorem 5.6 ([3, 17], cf. [2]). *Let G be a connected complex Lie group, H a closed complex Lie subgroup of G , and assume that G/H is compact. Denote by H° the connected component of H containing the identity element and N the normalizer of H° in G . Then N is a parabolic subgroup of G and N/H is a parallelizable complex manifold.*

Proof of Theorem 1.5 By Lemma 5.3 \mathfrak{g} is a semisimple Lie algebra of non-compact type. Let \mathfrak{k} be a maximal compact subalgebra of \mathfrak{g} containing \mathfrak{v} and $\mathfrak{g} = \mathfrak{k} + \mathfrak{q}$ a Cartan decomposition with respect to \mathfrak{k} . Let

$$(5.17) \quad \mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-k} + \cdots + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \cdots + \mathfrak{g}_k$$

be the decomposition induced from D . Note that $\mathfrak{g}_{\leq 0}$ is a Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

Without loss of generality, let us assume that V is connected. Let $G_{\leq 0}$ denote the connected subgroup corresponding to the Lie subalgebra $\mathfrak{g}_{\leq 0}$ in $G^{\mathbb{C}}$. Then $G^{\mathbb{C}}/G_{\leq 0}$ is a complex homogeneous manifold and there is a finite to one holomorphic immersion from $M = G/V$ into an open G -orbit on $G^{\mathbb{C}}/G_{\leq 0}$ given by $gV \mapsto gG_{\leq 0}$. Here the number of fiber elements is the number of connected components of $G \cap G_{\leq 0}$. Let G_u denote the compact real form of $G^{\mathbb{C}}$ with respect to G , i.e., $\mathfrak{g}_u = \mathfrak{k} + \sqrt{-1}\mathfrak{q}$. Then since G_u -orbit at

$x_0 \in G^{\mathbb{C}}/G_{\leq 0}$ with $x_0 = eG_{\leq 0}$ is open and closed, $G^{\mathbb{C}}/G_{\leq 0}$ is equal to $G_u/(G_u \cap G_{\leq 0})$. This implies that $G^{\mathbb{C}}/G_{\leq 0}$ is compact. By Theorem 5.6 there is a fibration

$$(5.18) \quad \pi : G^{\mathbb{C}}/G_{\leq 0} \rightarrow G^{\mathbb{C}}/P$$

where $P = N_{G^{\mathbb{C}}}(G_{\leq 0})$ is the normalizer of $G_{\leq 0}$ in $G^{\mathbb{C}}$ and P is a parabolic subgroup in $G^{\mathbb{C}}$.

Let \widetilde{M} be a flag domain that is an open G -orbit at $\pi(x_0)$ on $G^{\mathbb{C}}/P$. Then the restriction of π on M gives a fibration $\pi|_M : M \rightarrow \widetilde{M}$ over a flag domain and the fibers are parallelizable complex manifolds. \square

Corollary 5.7. *Under the same condition in Theorem 1.5, suppose that $n_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}_{\leq 0}) := \{x \in \mathfrak{g}^{\mathbb{C}} : [x, \mathfrak{g}_{\leq 0}] = 0\} = 0$. Then the universal covering of M is a flag domain.*

Corollary 5.8. *Under the same condition in Theorem 1.5, suppose that M is simply connected. Then the fiber is a complex torus.*

Let $\mathfrak{g} = \mathfrak{k}' + \mathfrak{q}'$ be a Cartan decomposition with the corresponding Cartan involution θ . It is known that θ is an inner automorphism of \mathfrak{g} if and only if \mathfrak{k}' contains a Cartan subalgebra of \mathfrak{g} . In this is the case, we say that the Lie algebra \mathfrak{g} and the Lie group G are of inner type. Note that \mathfrak{g} is of inner type if and only if all the simple ideals \mathfrak{g} are of inner type. The Cartan classification yields the following list of simple Lie algebras of inner type:

$$(5.19) \quad \begin{aligned} &\mathfrak{sl}_2(\mathbb{R}), \mathfrak{su}_{p,q}, \mathfrak{so}(p, q) \text{ (} p \text{ or } q \text{ even)}, \mathfrak{so}_{2n}^*, \mathfrak{sp}_{2n}(\mathbb{R}), \mathfrak{sp}_{p,q} \\ &E_{III}, E_{III}, EV, EVI, EVII, EVIII, EIX, FI, FII, G. \end{aligned}$$

Proof of Corollary 1.6 In [13], Mal'cev proved that if a real form G of inner type has an open orbit in $G^{\mathbb{C}}/L$ with a closed complex subgroup of $G^{\mathbb{C}}$, then L is a parabolic group of $G^{\mathbb{C}}$. Hence $G_{\leq 0}$ is a parabolic subalgebra of $G^{\mathbb{C}}$. Let σ denote the complex conjugate respect to \mathfrak{g} in $\mathfrak{g}^{\mathbb{C}}$. Then $\mathfrak{g}_{\leq 0} \cap \sigma(\mathfrak{g}_{\leq 0}) = \mathfrak{g}_0 = \mathfrak{v}^{\mathbb{C}}$ should contain a Cartan subalgebra of \mathfrak{g} (cf. [9] Corollary 2.1.3). \square

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